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Quaternion Dirac Equation and Supersymmetry

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Abstract Quaternion Dirac equation has been analyzed and its supersymmetrization has been discussed consistently. It has been shown that the quaternion Dirac equation automatically describes the spin structure with its spin up and spin down components of two component quaternion Dirac spinors associated with positive and negative energies. It has also been shown that the supersymmetrization of quaternion Dirac equation works well for different cases associated with zero mass, nonzero mass, scalar potential and generalized electromagnetic potentials. Accordingly we have discussed the splitting of supersymmetrized Dirac equation in terms of electric and magnetic fields.

Keywords Quaternion · Dirac-equation and supersymmetry

1 Introduction

Symmetries are one of the most powerful tools in theoretical physics. The two component formulation of complex numbers, and the non commutative algebra of quaternions, are possibly the two most important discoveries in mathematics. Quaternions were very first example of hyper complex numbers having the significant impacts on mathematics & physics [1]. Because of their beautiful and unique properties quaternions attracted many to study the laws of nature over the field of these numbers. Quaternions are already used in the context of special relativity [2], electrodynamics [3–5], Maxwell's equation [6], quantum mechanics [7, 8], Quaternion Oscillator [9], gauge theories [10–12], Supersymmetry [13] and many other branches of Physics [14] and Mathematics [15, 16]. On the other hand supersymmetry (SUSY) is described as the symmetry of bosons and fermions [17–19]. Gauge Hierarchy

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problem, not only suggests that the SUSY exists but put an upper limit on the masses of super partners [20, 21]. The exact SUSY implies exact fermion-boson masses, which has not been observed so far. Hence it is believed that supersymmetry is an approximate symmetry and it must be broken [22, 23]. We have considered following two motivations to study the higher dimensional supersymmetric quantum mechanics [24, 25] over the field of Quaternions.

1. Supersymmetric field theory can provide us realistic models of particle physics which do not suffer from gauge hierarchy problem and role of quaternions will provide us simplex and compact calculation accordingly.
2. Quaternions super symmetric quantum mechanics can give us new window to understand the behavior of supersymmetric partners and mechanism of super symmetry breaking etc.
3. Quaternions are capable to deal the higher dimensional structure and thus include the theory of monopoles and dyons [26, 27].

Keeping these facts in mind and to observe the role of quaternions in supersymmetry, we have developed the quaternion Dirac equation parallel to Dirac Pauli representation. It has been shown that the quaternion Dirac equation automatically describes the spin structure with its spin up and spin down components of two component quaternion Dirac spinors associated with positive and negative energies. We have obtained the free particle solutions of quaternion Dirac equation for quaternion, complex and real spinor representations. It has been shown that one component quaternion valued Dirac spinor is isomorphic to two component complex spinors or four components real representation. It has also been shown that the supersymmetrization of quaternion Dirac equation works well for different cases associated with zero mass, nonzero mass, scalar potential and generalized electromagnetic potentials. Accordingly we have discussed the splitting of supersymmetrized Dirac equation in terms of electric and magnetic fields. Accordingly, the super charges are calculated in all cases and it is shown that the Hamiltonian operator commutes with the super charges and the relations between the Schrödinger Hamiltonian and Dirac Hamiltonian are discussed in terms of super charges for different cases.

2 Definition

A quaternion ϕ is expressed as

$$\phi = e_0\phi_0 + e_1\phi_1 + e_2\phi_2 + e_3\phi_3, \tag{1}$$

where $\phi_0, \phi_1, \phi_2, \phi_3$ are the real quartets of a quaternion and e_0, e_1, e_2, e_3 are called quaternion units and satisfies the following relations,

$$\begin{aligned} e_0^2 &= e_0 = 1, \\ e_0 e_i &= e_i e_0 = e_i \quad (i = 1, 2, 3), \\ e_i e_j &= -\delta_{ij} + \varepsilon_{ijk} e_k \quad (i, j, k = 1, 2, 3), \end{aligned} \tag{2}$$

where δ_{ij} is the Kronecker delta and ε_{ijk} is the three index Levi-Civita symbols with their usual definitions. The quaternion conjugate $\bar{\phi}$ is then defined as

$$\bar{\phi} = e_0\phi_0 - e_1\phi_1 - e_2\phi_2 - e_3\phi_3. \tag{3}$$

Here ϕ_0 is real part of the quaternion defined as

$$\phi_0 = \mathrm{Re}\,\phi = \frac{1}{2}(\bar{\phi} + \phi). \tag{4}$$

If $\mathrm{Re}\,\phi = \phi_0 = 0$, then $\phi = -\bar{\phi}$ and imaginary ϕ is called pure quaternion and is written as

$$\mathrm{Im}\,\phi = e_1\phi_1 + e_2\phi_2 + e_3\phi_3. \tag{5}$$

The norm of a quaternion is expressed as

$$N(\phi) = \bar{\phi}\phi = \phi\bar{\phi} = \phi_0^2 + \phi_1^2 + \phi_2^2 + \phi_3^2 \geq 0 \tag{6}$$

and the inverse of a quaternion is described as

$$\phi^{-1} = \frac{\bar{\phi}}{|\phi|} \tag{7}$$

while the quaternion conjugation satisfies the following property

$$\overline{(\phi_1\phi_2)} = \bar{\phi}_2\bar{\phi}_1. \tag{8}$$

The norm of the quaternion (6) is positive definite and enjoys the composition law

$$N(\phi_1\phi_2) = N(\phi_1)N(\phi_2). \tag{9}$$

Quaternion (1) is also written as $\phi = (\phi_0, \vec{\phi})$ where $\vec{\phi} = e_1\phi_1 + e_2\phi_2 + e_3\phi_3$ is its vector part and ϕ_0 is its scalar part. The sum and product of two quaternions are

$$\begin{aligned}(\alpha_0, \vec{\alpha}) + (\beta_0, \vec{\beta}) &= (\alpha_0 + \beta_0, \vec{\alpha} + \vec{\beta}),\\ (\alpha_0, \vec{\alpha})(\beta_0, \vec{\beta}) &= (\alpha_0\beta_0 - \vec{\alpha}.\vec{\beta}, \alpha_0\vec{\beta} + \beta_0\vec{\alpha} + \vec{\alpha} \times \vec{\beta}).\end{aligned} \tag{10}$$

Quaternion elements are non-Abelian in nature and thus represent a division ring.

3 Quaternion Dirac Equation

Let us define respectively the space-time four vector, momentum four vector and four differential operator as quaternion in the following manner on using natural units $c = \hbar = 1$ and $i = \sqrt{-1}$ through out the text;

$$x = e_1x_1 + e_2x_2 + e_3x_3 + x_4 = -it + e_1x_1 + e_2x_2 + e_3x_3, \tag{11}$$

$$p = e_1p_1 + e_2p_2 + e_3p_3 + p_4 = -iE + e_1p_1 + e_2p_2 + e_3p_3, \tag{12}$$

$$\boxdot = e_1\partial_1 + e_2\partial_2 + e_3\partial_3 + \partial_4 = -i\frac{\partial}{\partial t} + e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} + e_3\frac{\partial}{\partial x_3}. \tag{13}$$

We may define accordingly the quaternion conjugate of these physical quantities as follows;

$$\overline{x} = -e_1x_1 - e_2x_2 - e_3x_3 + x_4 = -it - e_1x_1 - e_2x_2 - e_3x_3, \tag{14}$$

$$\overline{p} = -e_1p_1 - e_2p_2 - e_3p_3 + p_4 = -iE - e_1p_1 - e_2p_2 - e_3p_3, \tag{15}$$

$$\overline{\boxdot} = -e_1\partial_1 - e_2\partial_2 - e_3\partial_3 + \partial_4 = -i\frac{\partial}{\partial t} - e_1\frac{\partial}{\partial x_1} - e_2\frac{\partial}{\partial x_2} - e_3\frac{\partial}{\partial x_3}. \tag{16}$$

Using the quaternion multiplication rule (2) we may write the norm of the above quaternion valued four vectors as

$$x\overline{x}=\overline{x}x=x_1^2+x_2^2+x_3^2+x_4^2=x_1^2+x_2^2+x_3^2-t^2, \tag{17}$$

$$p\overline{p}=\overline{p}p=p_1^2+p_2^2+p_3^2+p_4^2=p_1^2+p_2^2+p_3^2-E^2, \tag{18}$$

$$\boxdot\overline{\boxdot}=\overline{\boxdot}\boxdot=\partial_1^2+\partial_2^2+\partial_3^2+\partial_4^2=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}-\frac{\partial^2}{\partial t^2}. \tag{19}$$

Equation (19) can also be related with the D'Alembertian operator in the fallowing manner i.e.

$$\Box=\boxdot\overline{\boxdot}=\overline{\boxdot}\boxdot=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}-\frac{\partial^2}{\partial t^2}=-\frac{\partial^2}{\partial t^2}+\nabla^2. \tag{20}$$

As such we may write the quaternion form of the Klein Gordon equation as follows,

$$(\Box-m^2)\phi_\alpha=(\boxdot\overline{\boxdot}-m^2)\phi_\alpha=(\overline{\boxdot}\,\boxdot-m^2)\phi_\alpha=0\quad(\alpha=0,1,2,3) \tag{21}$$

where ϕ_α are the components of quaternion scalar field (1). Now we may describe the quaternion form of Dirac equation from the following Schrodinger equation

$$\widehat{H_D}\psi=i\frac{\partial\psi}{\partial t} \tag{22}$$

where

$$\widehat{H_D}=\sum_{l=1}^{3}\alpha_l p_l+\beta m \tag{23}$$

and α and β are the arbitrary coefficients and thus satisfy the following properties in order to obtain the Klein Gordon equation (21) i.e.

$$\begin{aligned}\alpha_l^2=1;&\qquad \beta^2=1\quad(l=1,2,3);\\ \alpha_l\beta+\beta\alpha_l=0;&\qquad \alpha_l\alpha_m+\alpha_m\alpha_l=0\end{aligned} \tag{24}$$

along with the following properties to maintain the Hermiticity of Dirac Hamiltonian given by (22) i.e.

$$H_D^\dagger=H_D\Rightarrow\alpha_l^\dagger=\alpha_l;\qquad \beta^\dagger=\beta. \tag{25}$$

Thus, we may define 2×2 quaternion valued α and β matrices satisfying the properties (24), (25) as follows [28, 29] i.e.

$$\alpha_l=\begin{bmatrix}0 & ie_l\\ ie_l & 0\end{bmatrix};\qquad \beta=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \tag{26}$$

where the quaternion basis elements e_l satisfy the properties given by (2). We may now introduce the plain wave solution of Dirac equation

$$\left(\sum_{l=1}^{3}\alpha_l p_l+\beta m\right)\psi=E\psi \tag{27}$$

as

$$\psi(x,t) = \psi e^{ip_\mu x^\mu} \tag{28}$$

where ψ is the quaternion valued Dirac spinor given by

$$\psi = \psi_0 + e_1\psi_1 + e_2\psi_2 + e_3\psi_3 \tag{29}$$

and may be decomposed as

$$\psi = \begin{bmatrix} \psi_a \\ \psi_b \end{bmatrix} = \begin{bmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ -\psi_3 \end{bmatrix}$$

as the two or four components Dirac spinor associated with the symplectic representation of quaternion as $\psi = \psi_a + e_2\psi_b$ in terms of complex and accordingly $\psi_a = \psi_0 + e_1\psi_1$ and $\psi_b = \psi_2 - e_1\psi_3$ over the field of real number representations. In other words we can write one component quaternion valued Dirac spinor which is isomorphic to two component complex spinors or four components real representation. In describing the theory of Dirac equation it is customary to take the Dirac spinor as the four component spinors with complex coefficients. Hence in (29) the spinor ψ may be described as bi-quaternion valued where all its components i.e. $\psi_0, \psi_1, \psi_2, \psi_3$ are complex ones and the complex quantity $i = \sqrt{-1}$ commutes with all the quaternion basis elements $e_0 = 1, e_1, e_2, e_3$. Using (26)–(29) we get the following two equations as

$$\begin{aligned} E\psi_a &= m\psi_a + ie_l.p_l\psi_b; \\ E\psi_b &= -m\psi_a + ie_l.p_l\psi_b \end{aligned} \tag{30}$$

or

$$\begin{aligned} (E-m)\psi_a &= ie_l.p_l\psi_b; \\ (E+m)\psi_b &= ie_l.p_l\psi_a. \end{aligned} \tag{31}$$

As such we may obtain the following types of four spinor amplitudes: of Dirac spinors i.e.:

(i) One component spinor amplitudes:

$$\begin{aligned} \psi^1 &= \left(1 + e_2.\frac{ie_l p_l}{E+m}\right) \quad (Energy = +ive, spin = \uparrow); \\ \psi^2 &= \left(1 + e_2.\frac{ie_l p_l}{E+m}\right)e_1 \quad (Energy = +ive, spin = \downarrow); \\ \psi^3 &= \left(e_2 - \frac{ie_l p_l}{E+m}\right) \quad (Energy = -ive, spin = \uparrow); \\ \psi^4 &= \left(e_2 - \frac{ie_l p_l}{E+m}\right)e_1 \quad (Energy = -ive, spin = \downarrow). \end{aligned} \tag{32}$$

(ii) Two component spinor amplitudes:

$$\begin{aligned}
\psi^1 &= \begin{pmatrix} 1 \\ \frac{ie_l p_l}{E+m} \end{pmatrix} \quad (Energy = +ive, spin = \uparrow); \\
\psi^2 &= \begin{pmatrix} 1 \\ \frac{ie_l p_l}{E+m} \end{pmatrix} e_1 \quad (Energy = +ive, spin = \downarrow); \\
\psi^3 &= \begin{pmatrix} \frac{-ie_l p_l}{E+m} \\ 1 \end{pmatrix} \quad (Energy = -ive, spin = \uparrow); \\
\psi^4 &= \begin{pmatrix} \frac{-ie_l p_l}{E+m} \\ 1 \end{pmatrix} e_1 \quad (Energy = -ive, spin = \downarrow).
\end{aligned} \tag{33}$$

(iii) Four component spinor amplitudes are obtained by restricting the direction of propagation along Z-*axis* i.e. $p_x = P_y = 0$ (direction of propagation) and on substituting $e_l = -i\sigma_l$ and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

along with the usual definitions of spin up and spin down amplitudes of spin i.e.

$$\begin{aligned}
\psi^1 &= \begin{pmatrix} 1 \\ 0 \\ \frac{|\vec{p}|}{E+m} \\ 0 \end{pmatrix} \quad (Energy = +ive, spin = \uparrow); \\
\psi^2 &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\frac{|\vec{p}|}{E+m} \end{pmatrix} \quad (Energy = +ive, spin = \downarrow); \\
\psi^3 &= \begin{pmatrix} -\frac{|\vec{p}|}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad (Energy = -ive, spin = \uparrow); \\
\psi^4 &= \begin{pmatrix} 0 \\ \frac{|\vec{p}|}{E+m} \\ 0 \\ 1 \end{pmatrix} \quad (Energy = -ive, spin = \downarrow).
\end{aligned} \tag{34}$$

As such we may obtain the solution of quaternion Dirac equation in terms of one component quaternion, two component complex and four component real spinor amplitudes. Equation (34) is same as obtained in the case of usual Dirac equation. Thus we may interpret that the $N = 1$ quaternion spinor amplitude is isomorphic to $N = 2$ complex and $N = 4$

real spinor amplitude solution of Dirac equation. We can accordingly interpret the minimum dimensional representation for Dirac equation is $N=1$ in quaternionic case, $N=2$ in complex case and $N=4$ for real number field.

4 Super-Symmetrization of Quaternion Dirac Equation

The quaternion free particle Dirac equation given by (27) may now be directly written in following covariant form,

$$i\gamma_\mu \partial_\mu \psi(x,t) = m\psi(x,t) \quad (\mu = 0,1,2,3) \tag{35}$$

where we have defined the following representation of gamma matrices in terms of quaternion basis elements i.e.

$$\begin{aligned} \gamma_0 &= \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \\ \gamma_l &= \beta\alpha_l = \begin{bmatrix} 0 & ie_l \\ -ie_l & 0 \end{bmatrix} \quad (l = 1,2,3) \end{aligned} \tag{36}$$

along with the following properties associated therein i.e.

$$\begin{aligned} \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu &= -2g_{\mu\nu}, \\ g_{\mu\nu} &= (-1,+1,+1,+1) \quad \forall \mu = \nu = 0,1,2,3, \\ g_{\mu\nu} &= 0 \quad \forall \mu \neq \nu. \end{aligned} \tag{37}$$

Let us discuss the super-symmetrization in following three cases.

Case I: For mass less free particle i.e. $m=0$ and external potential $\Phi = 0$; (35) becomes

$$i\gamma_\mu \partial_\mu \psi(x,t) = 0. \tag{38}$$

Let us consider the following solutions of this equation as

$$\psi(x,t) = \psi(x) e^{i(\vec{p}\cdot\vec{x} - Et)}. \tag{39}$$

Thus (35) reduces to

$$(\gamma_0 E - \gamma_1 p_1 - \gamma_2 p_2 - \gamma_3 p_3)\psi(x) = 0 \tag{40}$$

which takes the following form

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} E - e_l \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} p_l \right\} \begin{Bmatrix} \psi_a \\ \psi_b \end{Bmatrix} = 0 \tag{41}$$

where $\psi_a = \psi_0 + e_1\psi_1$ and $\psi_b = \psi_2 - e_1\psi_3$. We thus obtain the following coupled equations

$$\begin{aligned} \widehat{A}\psi_a(x) &= E\psi_b(x), \\ \widehat{A}^\dagger \psi_b(x) &= E\psi_a(x) \end{aligned} \tag{42}$$

where $\widehat{A} = \widehat{A}^{\dagger} = ie_l\widehat{p_l}$. We can now decouple (42) as

$$\begin{aligned} \widehat{A}\widehat{A}^{\dagger}\psi_b(x) &= E^2\psi_b(x), \\ \widehat{A}^{\dagger}\widehat{A}\psi_a(x) &= E^2\psi_a(x) \end{aligned} \tag{43}$$

which gives rise to a single supersymmetric decoupled equation as

$$P_l^2\psi_{a,b}(x) = E^2\psi_{a,b}(x) \tag{44}$$

where $\psi_a(x)$ and $\psi_b(x)$ are eigen functions of partner Hamiltonians $H_- = \widehat{A}^{\dagger}\widehat{A}$ and $H_+ = \widehat{A}\widehat{A}^{\dagger}$. The supersymmetric Hamiltonian is thus described as

$$\widehat{H} = \begin{bmatrix} H_+ & 0 \\ 0 & H_- \end{bmatrix} = \begin{bmatrix} P_l^2 & 0 \\ 0 & P_l^2 \end{bmatrix} = \hat{H}_D^2 \tag{45}$$

where $\hat{H}_D$ is the Dirac Hamiltonian given by

$$\hat{H}_D = \sum_{l=1}^{3}\alpha_l p_l = \begin{bmatrix} 0 & ie_l p_l \\ ie_l p_l & 0 \end{bmatrix} \tag{46}$$

and can be compared with the Dirac Hamiltonian given by Thaller [30, 31] as

$$\hat{H}_D = \begin{bmatrix} M_+ & Q_D^{\dagger} \\ Q_D & -M_- \end{bmatrix} \tag{47}$$

where we have obtained $M_+ = M_- = 0$ and $Q_D = Q_D^{\dagger} = ie_l p_l$ along with the following supersymmetric conditions

$$Q_D^{\dagger}M_- = M_+Q_D^{\dagger}; \qquad Q_D M_+ = M_- Q_D. \tag{48}$$

Restricting the propagation along x-axis to discuss the quantum mechanics in two dimensional space time, let us write

$$\begin{aligned} \widehat{p_l} &= -i\frac{d}{dx}, \\ ie_l\widehat{p_l} &= e_2\frac{d}{dx}, \\ \widehat{H} &= \begin{bmatrix} -\frac{d^2}{dx^2} & 0 \\ 0 & -\frac{d^2}{dx^2} \end{bmatrix} \end{aligned} \tag{49}$$

or

$$\widehat{H} = \begin{bmatrix} \widehat{Q}\widehat{Q}^{\dagger} & 0 \\ 0 & \widehat{Q}^{\dagger}\widehat{Q} \end{bmatrix} \tag{50}$$

where supercharges are described in terms of quaternion units i.e.

$$\widehat{Q} = \begin{bmatrix} 0 & -e_2^\dagger \frac{d}{dx} \\ 0 & 0 \end{bmatrix},$$
$$\widehat{Q}^\dagger = \begin{bmatrix} 0 & 0 \\ e_2 \frac{d}{dx} & 0 \end{bmatrix}. \tag{51}$$

As such we may obtain the supersymmetry algebra as

$$\begin{aligned} [\widehat{Q}, \widehat{H}] &= [\widehat{Q}, \widehat{H}^\dagger] = 0, \\ \{\widehat{Q}, \widehat{Q}\} &= \{\widehat{Q}^\dagger, \widehat{Q}^\dagger\} = 0, \\ \{\widehat{Q}, \widehat{Q}^\dagger\} &= \widehat{H}. \end{aligned} \tag{52}$$

Here $\widehat{Q}^\dagger$, converts the upper component spinor $\left\{ {\psi_a \atop 0} \right\}$ to a lower one $\left\{ {0 \atop \psi_b} \right\}$ and similarly $\widehat{Q}$ converts the lower component Spinor $\left\{ {0 \atop \psi_b} \right\}$ to upper one $\left\{ {\psi_a \atop 0} \right\}$. If ψ to be an eigen state of H_+ (H_-), $\widehat{Q}\psi$ $(\widehat{Q}^\dagger\psi)$ is the eigen state of that of H_+ (H_-) with equal energy.

Case II: $m \neq 0$ but constant and potential $\Phi = 0$.

Corresponding Dirac's equation (35) with its solution (39) for this case is described as

$$(\gamma_0 E - \gamma_1 p_1 - \gamma_2 p_2 - \gamma_3 p_3 - m)\psi(x) = 0. \tag{53}$$

Substituting γ_0 and γ_l from (36) into (53) we get following coupled (supersymmetric) equations

$$\begin{aligned} i e_l p_l \psi_a &= \widehat{A}\psi_a = (E + m)\psi_b, \\ i e_l p_l \psi_b &= \widehat{A}^\dagger \psi_b = (E - m)\psi_a \end{aligned} \tag{54}$$

which leads to following sets of decoupled equations,

$$\begin{aligned} \widehat{A}^\dagger \psi_b &= (E - m)\psi_a, \\ \widehat{A}\psi_a &= (E + m)\psi_b, \\ \widehat{A}^\dagger \widehat{A}\psi_a &= H_-\psi_a = P_l^2 \psi_a = (E^2 - m^2)\psi_a, \\ \widehat{A}\widehat{A}^\dagger \psi_b &= H_-\psi_b = P_l^2 \psi_b = (E^2 - m^2)\psi_b, \\ \widehat{P_l}^2 \psi_{a,b} &= (E^2 - m^2)\psi_{a,b} \end{aligned} \tag{55}$$

which are the Schrödinger equation for free particle. We may now write the Schrödinger Hamiltonian and Schrödinger charges as

$$\widehat{H_s} = \begin{bmatrix} P_l^2 & 0 \\ 0 & P_l^2 \end{bmatrix}; \qquad \widehat{Q_s} = \begin{bmatrix} 0 & i e_l p_l \\ 0 & 0 \end{bmatrix}; \qquad \widehat{Q_s}^\dagger = \begin{bmatrix} 0 & 0 \\ i e_l p_l & 0 \end{bmatrix}, \tag{56}$$

where $\widehat{H_s}$, $\widehat{Q_s}$ and $\widehat{Q_s}^\dagger$ satisfy the SUSY algebra given by (52). Correspondingly the Dirac Hamiltonian is described as

$$\widehat{H_D} = \sum_{l=1}^{3} \alpha_l p_l + m = \begin{bmatrix} m & i e_l p_l \\ i e_l p_l & -m \end{bmatrix}, \tag{57}$$

where $M_+ = M_- = m$ and $\widehat{Q_D} = \widehat{Q_D}^\dagger = ie_l p_l$ and hence $\widehat{Q_D}, \widehat{Q_D}^\dagger$, M_+ and M_- satisfy the SUSY algebra given by (52). Thus the SUSY Hamiltonian resembles with the square of Dirac Hamiltonian described as

$$\widehat{H} = \widehat{H}_D^2 = \widehat{H}_s + m^2 = \begin{bmatrix} \widehat{Q}\widehat{Q}^\dagger & 0 \\ 0 & \widehat{Q}^\dagger\widehat{Q} \end{bmatrix} = \begin{bmatrix} \widehat{P_l}^2 + m^2 & 0 \\ 0 & \widehat{P_l}^2 + m^2 \end{bmatrix}. \tag{58}$$

Similarly, we get the following relations while restricting ourselves for two dimensional structure of space and time i.e.

$$\widehat{H} = \begin{bmatrix} \widehat{Q}\widehat{Q}^\dagger & 0 \\ 0 & \widehat{Q}^\dagger\widehat{Q} \end{bmatrix} = \begin{bmatrix} -\frac{d^2}{dx^2} + m^2 & 0 \\ 0 & -\frac{d^2}{dx^2} + m^2 \end{bmatrix} \tag{59}$$

where

$$\begin{aligned} \widehat{Q} &= \begin{bmatrix} 0 & -e_2^\dagger \frac{d}{dx} + m \\ 0 & 0 \end{bmatrix}, \\ \widehat{Q}^\dagger &= \begin{bmatrix} 0 & 0 \\ -e_2^\dagger \frac{d}{dx} + m & 0 \end{bmatrix}. \end{aligned} \tag{60}$$

Hence we have restored the property of SUSY quantum mechanics and obtain the commutation and anticommutation relations similar to that of (52) for the free particle Dirac equation with mass as well.

Case III: Mass $m \neq 0$ but not constant and is space-time dependent i.e. $m = m(x)$ and potential $\Phi = 0$.

Let us discuss and verify the SUSY quantum mechanics for $m \neq 0$ i.e. $m = m(x)$ with scalar potential $\Phi = 0$. We extend the present theory in the same manner and express Dirac Hamiltonian in the following form:

$$\widehat{H_D} = \begin{bmatrix} 0 & ie_l p_l - im \\ ie_l p_l - im & 0 \end{bmatrix} \tag{61}$$

where $M_+ = M_- = 0$, $\widehat{Q_D} = ie_l p_l + im$, $\widehat{Q_D}^\dagger = ie_l p_l - im$ and again $\widehat{Q_D}, \widehat{Q_D}^\dagger$, M_+ and M_- satisfy the SUSY algebra given by (52). Thus operating Dirac Hamiltonian given by (61) on the Dirac Spinor in the manner $\widehat{H_D}\psi = E\psi$ we get the following coupled (supersymmetric) differential equations i.e.

$$\begin{aligned} (ie_l p_l + im)\psi_a &= \widehat{A}^\dagger \psi_a = E\psi_b, \\ (ie_l p_l - im)\psi_b &= \widehat{A}\psi_b = E\psi_a, \end{aligned} \tag{62}$$

where $\widehat{A}^\dagger = (ie_l p_l + im)$ and $\widehat{A} = (ie_l p_l - im)$ and hence we get the following form of Schrödinger Hamiltonian i.e.

$$\widehat{H_S} = \begin{bmatrix} (ie_l p_l - im)(ie_l p_l + im) & 0 \\ 0 & (ie_l p_l - im)(ie_l p_l - im) \end{bmatrix} = \widehat{H_D}^2 \tag{63}$$

which may be written as

$$\widehat{H_S} = \begin{bmatrix} \widehat{Q_s}\widehat{Q}_S^\dagger & 0 \\ 0 & \widehat{Q_S}^\dagger \widehat{Q_S} \end{bmatrix} = \widehat{H_D}^2 \tag{64}$$

with the following representation of super charges,

$$\widehat{Q_s} = \begin{bmatrix} 0 & (ie_l p_l - im) \\ 0 & 0 \end{bmatrix},$$
$$\widehat{Q_s}^{\dagger} = \begin{bmatrix} 0 & 0 \\ (ie_l p_l + im) & 0 \end{bmatrix}. \tag{65}$$

Equations (54) to (65) satisfy the supersymmetric quantum mechanical relations given by (47) and as such the supersymmetry.

Case IV: Dirac equation in electromagnetic field. Before writing the quaternion Dirac equation in generalized electromagnetic fields of dyons let us start with the quaternion gauge transformations. A Q-field (1) is described in terms of following SO(4) local gauge transformations [7, 10, 11]

$$\phi \to \phi' = U\phi\bar{V}U, V\varepsilon Q, U\bar{U} = V\overline{V} = 1. \tag{66}$$

The covariant derivative for this is then written in terms of two gauge potentials as

$$D_\mu\phi = \partial_\mu\phi + A_\mu\phi - \phi B_\mu \tag{67}$$

where potential transforms as

$$A'_\mu = UA_\mu\bar{U} + U\partial_\mu\bar{U},$$
$$B'_\mu = VB_\mu\bar{V} + V\partial_\mu\bar{V} \tag{68}$$

and

$$\bar{\phi}'\phi' = \overline{(U\phi\bar{V})}((U\phi\bar{V}) = \bar{\phi}\phi = \phi_0^2 + |\vec{\phi}|^2. \tag{69}$$

Here we identify the non-Abelian gauge fields A_μ and B_μ as the gauge potentials respectively for electric and magnetic charges of dyons [12, 14, 26, 27]. Corresponding field momentum of (67) may also be written as follows

$$P_\mu\phi = p_\mu\phi + A_\mu\phi - \phi B_\mu \tag{70}$$

where the gauge group $SO(4) = SU(2)_e \times SU(2)_g$ is constructed in terms of quaternion units of electric and magnetic gauges. Accordingly, the covariant derivative thus describes two different gauge field strengths i.e.

$$[D_\mu, D_\nu]\phi = f_{\mu\nu}\phi - \phi h_{\mu\nu},$$
$$f_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} + [A_\mu, A_\nu],$$
$$h_{\mu\nu} = B_{\mu,\nu} - B_{\nu,\mu} + [B_\mu, B_\nu] \tag{71}$$

where $f_{\mu\nu}$ and $h_{\mu\nu}$ are gauge field strengths associated with electric and magnetic charges of dyons respectively.We may now write the Dirac equation (34) as

$$i\gamma_\mu D_\mu\psi(x,t) = m\psi(x,t). \tag{72}$$

Following some restrictions and using the properties of quaternions we may write the Dirac equation (72) as

$$\begin{bmatrix} m & ie_l(p_l + A_l) \\ -ie_l(p_l + A_l) & -m \end{bmatrix} \begin{bmatrix} \psi_a \\ \psi_b \end{bmatrix} + \begin{bmatrix} \psi_a \\ \psi_b \end{bmatrix} \begin{bmatrix} 0 & -ie_l B_l \\ ie_l B_l & 0 \end{bmatrix}$$
$$= (E - A_0 + B_0) \begin{bmatrix} \psi_a \\ \psi_b \end{bmatrix} \tag{73}$$

where $\varphi_a = \varphi_0 + e_1\varphi$ and $\varphi_b = \varphi_2 - e_1\varphi_3$. As such we obtain the following set of coupled equations responsible for the supersymmetric realization of the theory i.e.

$$\begin{aligned} E\psi_a &= (m + A_0 - B_0)\psi_a + ie_l(p_l + A_l)\psi_b - i\psi_b e_l B_l, \\ E\psi_b &= (-m + A_0 - B_0)\psi_b - ie_l(p_l + A_l)\psi_a + i\psi_a e_l B_l, \\ ie_l(p_l + A_l)\psi_b - i\psi_b e_l B_l &= (E - A_0 + B_0 - m)\psi_a, \\ -ie_l(p_l + A_l)\psi_a + i\psi_a e_l B_l &= (E - A_0 + B_0 + m)\psi_b, \\ \widehat{A}^\dagger\psi_b &= (E - m - A_0 + B_0)\psi_a = ie_l(p_l + A_l)\psi_b - i\psi_b e_l B_l, \\ \widehat{A}\psi_a &= (E + m - A_0 + B_0)\psi_b = -ie_l(p_l + A_l)\psi_a + i\psi_a e_l B_l, \\ \widehat{A}^\dagger\widehat{A}\psi_a &= (E^2 - m^2)\psi_a, \\ \widehat{A}\widehat{A}^\dagger\psi_b &= (E^2 - m^2)\psi_b \end{aligned} \tag{74}$$

where we have restricted ourselves to the case of two dimensional supersymmetry by imposing the condition $A_1^\dagger = -A_1$, $A_2^\dagger = -A_2$, $A_3^\dagger = -A_3$, $B_1^\dagger = -B_1$, $B_2^\dagger = -B_2$, $B_3^\dagger = -B_3$ along with other subsidiary conditions to restore the supersymmetry. As such, it is possible to supersymmetrize the Dirac equation for generalized electromagnetic fields of dyons and we may obtain the commutation and anticommutation relations given by (47) to verify the supersymmetric quantum mechanics in this case.

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